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Relation of Spatial Skills to Calculus Proficiency: A Brief Report

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ABSTRACT

Spatial skills have been shown in various longitudinal studies to be related to multiple science, technology, engineering, and math (STEM) achievement and retention. The specific nature of this relation has been probed in only a few domains, and has rarely been investigated for calculus, a critical topic in preparing students for and in STEM majors and careers. We gathered data on paper-and-pencil measures of spatial skills (mental rotation, paper folding, and hidden figures); calculus proficiency (conceptual knowledge and released Advanced Placement [AP] calculus items); coordinating graph, table, and algebraic representations (coordinating multiple representations); and basic graph/table skills. Regression analyses suggest that mental rotation is the best of the spatial predictors for scores on released AP calculus exam questions ($\beta = 0.21$), but that spatial skills are not a significant predictor of calculus conceptual knowledge. Proficiency in coordinating multiple representations is also a significant predictor of both released AP calculus questions ($\beta = 0.37$) and calculus conceptual knowledge ($\beta = 0.47$). The spatial skills tapped by the measure for mental rotation may be similar to those required to engage in mental animation of typical explanations in AP textbooks and in AP class teaching as tested on the AP exam questions. Our measure for calculus conceptual knowledge, by contrast, did not require coordinating representations.

Scores on spatial skills measures are predictive of success in multiple science, technology, engineering, and math (STEM) tasks, majors, and careers (Harle & Towns, 2010; Höffler, 2010; Sorby, 2009; Wai, Lubinski, Benbow, & Steiger, 2010). Why are spatial skills related to STEM? For some STEM fields, the connections are obvious—structural geology involves the study of deformations in three dimensions, chemistry involves reasoning about interactions among electrons that take a spatial configuration around atomic nuclei. For many STEM fields, however, the relative importance of different spatial skills, and the relation of spatial skills to different skills in the domain have been underexplored (Hegarty, Crookes, Dara-Abrams, & Shipley, 2010). One such under-researched area of STEM is mathematics, and the key course calculus, in particular.

The role of spatial skills in calculus proficiency might be explained by the spatial nature of Cartesian graphs, which are the most frequently used visualization in calculus teaching and textbooks (Chang, Tran, & Cromley, 2016). For example, Bektasli (2006) found a significant relation between spatial skills and graph skills specific to interpreting slope ($r = 0.28$ for simpler spatial problems on the Purdue Spatial Visualizations Test (PSVT) and 0.40 for two-step problems on the PSVT; the PSVT is a 3D mental rotations measure very similar to the Mental Rotations Test we used (described next). The spatial skills-calculus proficiency relation would also be expected because

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central calculus tasks are inherently spatial; for example, visualizing the slope of a tangent to a curve as x values change for the derivative or imagining accumulating the area of “slices under a curve” for a range of x values for integration (see Bremigan, 2005; Sorby, Casey, Veurink, & Dulaney, 2013; Zimmerman, 1991).

Coordinating multiple mathematical representations and spatial skills

Decades of work by mathematics education researchers have documented that mathematics teachers can foster student understanding through developing students’ proficiency using and coordinating multiple representations (Bell & Janvier, 1981; Brenner et al., 1997; Hiebert & Carpenter, 1992; Kaput, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Roschelle et al., 2010; Yerushalmy, 1991). In fact, mathematical *understanding* is often defined in terms of fluency connecting representations. For example, the US National Research Council report *Adding it Up* stated:

> A significant indicator of conceptual understanding is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes. To find one’s way around the mathematical terrain, it is important to see how the various representations connect with each other, how they are similar, and how they are different. The degree of students’ conceptual understanding is related to the richness and extent of the connections they have made. (National Research Council, 2001, p. 119)

In many cases, mathematical problem solving draws heavily on spatial skills (e.g., Ganley & Vasilyeva, 2011). However, there has been little empirical work investigating relations between spatial skills and calculus proficiency. Samuels (2010) administered the “*Development*” and “*Rotations*” subscales from the Purdue Spatial Visualization Test and had students solve calculus problems, which were coded for proficiency in two subscales—limits and the derivative. Using these measures of spatial skill, the only significant correlation was between *Development* and scores on calculus problems involving the derivative, $r = 0.53$.

Constituent spatial skills

Although there is not clear agreement about constituent spatial skills, evidence suggests that these skills may be considered as varying on multiple dimensions. One challenge in assessing the effects of spatial skills is therefore the selection of appropriate spatial measure(s). In one typology, two important dimensions are reasoning about spatial relations within an object—intrinsic—versus between two objects—extrinsic—and reasoning about static objects versus moving (or dynamic) objects (Chatterjee, 2008; Newcombe & Shipley, 2015; Uttal et al., 2013). In the present research, we gave measures of static and dynamic reasoning with three different spatial skills measures—the Mental Rotations Test (Peters et al., 1995; an intrinsic and dynamic spatial measure that taps mental rotations), the Hidden Figures Test (Ekstrom, French, Harman, & Derman, 1976; an intrinsic and static spatial measure that taps disembedding), and the Paper Folding Test (Ekstrom et al., 1976; an intrinsic and extrinsic dynamic spatial measure that taps spatial visualization).

In addition, spatial skills are malleable (Sorby, 2009; Uttal et al., 2013), which means that finding a relation between spatial skills and calculus proficiency would suggest an avenue for improving calculus proficiency at the high school level. Developing spatial skills could potentially keep talented students in the STEM pipeline and perhaps decrease the need for undergraduate mathematics remediation. In a meta-analysis of spatial training studies, Uttal and colleagues found large effects of training on the three spatial measures we use in the present study: the effect of spatial training on scores on the Hidden Figures measure was Hedges’ $g = 0.48$, on the Paper Folding measure was $g = 0.65$, and on the Mental Rotations measure was $g = 0.82$. Specifically in the domain of calculus, Sorby and colleagues (2013) found that practice reasoning about 3D objects and rotation improved calculus grades in low-spatial engineering students ($d = 0.20$).

Other predictors of calculus proficiency

In order to understand the unique contribution of spatial skills, it is important to control for other predictors of calculus proficiency. Two major recent changes to the US secondary and undergraduate calculus curriculum are: (1) an emphasis on conceptual understanding of calculus instead of manipulation of equations (Steen, 1988), and (2) requiring students to work with multiple representations of the same function, such as a graph, a data table, a formula, and a text passage (College Board, 2012; Hughes-Hallett et al., 2010). Here, we briefly review research on (1) representational familiarity (basic graph/table skills), and (2) coordinations among algebraic, tabular, and graphical representations (types of coordinating multiple representations or CMR).

Representational familiarity

Although most students gain familiarity with algebraic representations in school mathematics, graph and table representations can be more or less familiar to learners, and are known to pose certain difficulties. In terms of graphs, younger students fall into the "graph as picture" fallacy (Bell & Janvier, 1981; Dugdale, 1993) and even undergraduate students have difficulty constructing accurate graphs (Geiger, Stradtmann, Vogel, & Seufert, 2011). Many textbook examples provide a function in algebra-symbolic notation with pairs of values to plot, encouraging students to think of graphing in terms of plotting points rather than plotting functions (Elia, Panaoura, Eracleous, & Gagatsis, 2007; Elia, Panaoura, Gagatsis, Gravvani, & Spyrou, 2008; Even, 1998; Monoyiou & Gagatsis, 2008). In addition, Knuth (2000) found that students tended to ignore the graph they were given to solve an algebra problem, and instead defaulted to algebraic manipulation (also see Eisenberg & Dreyfus, 1991). In terms of tables, students show low performance on both experimental tasks tying tables to formulas as well as on the low-stakes US National Assessment of Educational Progress (NAEP) tests. Blanton and Kaput (2005) found that students might read down the y values in a table and fail to recognize that changes in the y-coordinate correspond to changes in the x-coordinate. Similarly, Lobato, Ellis, and Munoz (2003) found that students struggled to relate x and y values when the rows in a table did not have consistent differences between x values. There is also evidence that graph and table skills are related to spatial skills as measured by the *Hidden Figures Test* (Ekstrom et al., 1976), $r = 0.30–0.51$ (Linn & Pulos, 1983).

Learning from multiple representations

The task of coordinating multiple representations (CMR) such as equations and graphs is a complex one. By coordinating multiple representations, we mean "The ability to coordinate the translation and switching between representations within the same domain" (Chang et al., 2016, p. 2). Coordinating multiple representations requires *matching* information from one representation to another (e.g., finding the x-intercepts in a graph of a quadratic function and locating them in a table), *comparing* the information in the two representations (e.g., determining how a linear slope is represented differently in an equation and in a graph; how a y-intercept is represented differently in an equation and in a graph), and often bringing prior knowledge to bear to make *inferences* that yield a coherent mental model that integrates the two sources. In some cases, CMR tasks require recognizing matches between representations (a receptive task) and in other cases, the learner is required to construct a graph, equation, or other representation (a productive task; Ainsworth, 2006). For example, Hitt (1998) found that both secondary mathematics teachers and their students had difficulty creating a graph representation from a pictorial representation.

Many theoretical lenses have been brought to bear on the problem of CMR in mathematics, including scheme (in the Piagetian sense) for change and how to represent change (e.g., Dorko & Weber, 2014); semiotic perspectives on using and learning to use various representations of function (e.g., Roth, 2003); and commognition (communication + cognition, Sfard, 2008) perspectives on learning from discourse practices (e.g., Park, 2015). This work examines how constituent skills (e.g., spatial skills) and performance on a CMR task connected to calculus proficiency. Our initial plan was to investigate differences between expert and novice CMR skills, and to correlate those differences with spatial skills (this expert-novice difference did not emerge as expected, a point discussed in more detail next). Thus, we used an empirical approach to identify which spatial skills are most closely related to success on CMR tasks.

One unique feature of CMRs among graphs, tables, and algebraic expressions is that for an experienced user, a direct correspondence can be made across representations (i.e., high overlap of information), although scholars have argued that the conceptual affordances of each representation are different (Moschkovich, Schoenfeld, & Arcavi, 1993). For example, with some background knowledge a student can find the y intercept in an algebraic expression of the form $y = f(x)$ with very little effort. The y-intercept is also relatively easy to find by scanning up and down the y axis in a graph. By contrast, finding the x intercept from an algebraic expression in the form $y = f(x)$ typically requires some calculation whereas the x intercept(s) can be found perceptually by scanning across the x axis in an appropriately scaled graph.

Measuring calculus proficiency

Like other mathematical subdomains, proficiency in calculus requires both procedural and conceptual knowledge. As with spatial skills, one challenge in assessing effects on proficiency is a precise definition of conceptual knowledge. Conceptual knowledge is often thought to reflect "an integrated and functional grasp of mathematical ideas" (National Research Council, 2001, p. 118). Consistent with this and other research on learning in mathematics, we view conceptual knowledge as recognizing and understanding the critical principles and features of problems within a domain as well as connections between different knowledge components in the domain (Booth, 2011; Hiebert & Wearne, 1996; Rittle-Johnson & Star, 2007). In contrast, we view procedural knowledge as being able to carry out known steps in order to solve a problem (Rittle-Johnson, Siegler, & Alibali, 2001). As noted, the nature of the tasks involved in learning calculus has changed to include more conceptual understanding (Hughes-Hallett et al., 2010), but research on conceptual understanding of calculus and of coordinating multiple visual representations has not kept pace.

We know little about the relative contribution of conceptual understanding of calculus or coordinating multiple representations as predictors of calculus proficiency (as measured by both problem solving and conceptual knowledge). In the present research, we used the spatial skills measures and a measure of coordinating multiple representations, together with a measure of representational familiarity (basic graph/table skills) to predict scores on a standardized calculus proficiency measure and a newly-developed measure of conceptual (i.e., noncomputational) understanding of calculus.

Research question

Based on these findings from the literature, our research question is as follows: What is the relation of spatial skills to calculus proficiency, after taking into account representational familiarity and CMR scores? More specifically, we ask which of three types of spatial skills are related to which of two aspects of calculus proficiency—scores on released AP calculus items and calculus conceptual knowledge.

Method

Participants

Participants were 77 calculus and pre-calculus students from two suburban high schools (n = 66) and engineering undergraduates (n = 11) from one large urban university in the US mid- Atlantic region. While we originally recruited the undergraduate students because we expected them to show higher calculus proficiency, exploratory analyses showed no significant difference between the high school and undergraduate students on any variable, so we collapsed the groups.

Measures

Measures included three standardized spatial skills measures, standardized measures of basic graph/table skills and coordinating multiple representations, and two measures of calculus proficiency: one researcher-constructed calculus conceptual knowledge measure and a measure made up of released items from the AP Calculus AB™ exam. For scoring, missed questions were marked as incorrect; an analysis of missing items showed that 87% of students tried to answer at least one item within each question stem, suggesting that missingness was not due to time limits. Obtained reliability with our sample is shown in Table 1.

Spatial skills

Mental rotations test. We administered the first 12 items of the Mental Rotations Test (*MRT-A*, in a CAD-redrawn version of Vandenberg & Kuse from Peters et al., 1995). The *MRT-A* is a well-known correlate of diagram comprehension. Participants are asked to match a “target” 3D figure with four other figures, two of which are rotated versions of the target and two of which are not. Students were given 3 minutes to complete the measure.

Paper folding test. We used the first 10 items of the Paper Folding Test (PFT; Ekstrom et al., 1976), a three-minute paper-and-pencil measure of spatial visualization from Educational Testing Service (ETS). In this test, participants see drawings of a square sheet of paper with one to three folds made in it. The last drawing has a hole punched in it, and participants are asked to identify which of five choices would match the hole-punched drawing if the sheet of paper were unfolded.

Hidden figures test. We administered the first 10 items of the Hidden Figures Test (Ekstrom et al., 1976), another three-minute paper-and-pencil measure of spatial visualization published by ETS. In this test, participants see a complex two-dimensional geometrical figure with simple figures embedded within it. The task is to identify which of a series of simple figures is found in each complex item.

Graph/table skills

We created a measure comprised of 6 released NAEP Grade 12 graph items and 5 released NAEP, National Assessment of Adult Literacy (NAAL), and Assessment of Literacy and Language (ALL) table items. These are from the “Easy” groups of multiple-choice items, which tap basic graph and table comprehension, such as finding a single data point/cell. Participants were given six minutes to complete the measure; for this measure and all researcher-developed measures presented below, the amount of time was sufficient for completing the items based on work with pilot participants.

Table 1. Correlations among and descriptive statistics on all variables.

Measure (Max)	1	2	3	4	5	6	7
1. AP (11)				N = 77			
2. MRT (12)	0.39						
3. PFT (10)	0.20	0.29					
4. HFT (10)	0.27	0.21	0.30				
5. CCM (32)	0.62	0.27	0.20	0.22			
6. CMR (8)	0.49	0.41	0.24	0.21	0.51		
7. GT (11)	0.11	0.19	0.18	0.14	0.09	0.18	
M	3.18	4.90	7.01	1.95	15.77	4.03	8.17
SD	2.71	2.98	2.05	1.61	5.99	2.05	1.42
Cronbach’s α	0.94	0.82	0.70	0.87	0.99	0.76	0.67

Note. All correlations with absolute value > 0.19 are statistically significant at $p < 0.05$. AP, released Advanced Placement calculus test questions; MRT, Mental Rotations Test; PFT, Paper Folding Test; HFT, Hidden Figures Test; CCM, Calculus Conceptual Measure; CMR, Coordinating Multiple Representations; GT, Graph/Table comprehension.

Coordinating multiple representations

We used the eight-item "Understand function representations" subscale of the multiple-choice Pre-Calculus Concept Assessment (Carlson, Oehrtman, & Engelke, 2010) to measure CMR skills. The measure has been well validated with students in middle school through college (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson et al., 2010; Oehrtman, Carlson, & Thompson, 2008). Participants were given five minutes to complete the measure.

Released AP calculus exam items

As one measure of calculus proficiency, we compiled 11 multiple-choice released AP Calculus AB™ and BC™ exam questions. Since these are well-validated questions used by The College Board for years before public release, they were expected to show excellent reliability and validity with the study sample. Students were given 15 minutes to complete the measure.

Calculus conceptual knowledge

As a second measure of calculus proficiency, we created 32 calculus conceptual knowledge items (presented in full in the Appendix). These measured students' understanding of functions and limits (Lauten, Graham, & Ferrini-Mundy, 1994), derivatives (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997), and the chain rule (Clark et al., 1997), which have been identified in the literature as core topics in a first calculus course. Each of the multiple-choice items on the conceptual test measured relationships and did not require any calculations. For example, one item asked participants to identify which information they would use to determine intervals on which the function f is increasing. Students responded by circling any combination of the following options; f, f', and f''. To ensure that the scale was unidimensional, we analyzed the scores with Velicer's (Velicer, Eaton, & Fava, 2000) Minimum Average Partial procedure, which suggested that a one-factor solution (first eigenvalue = 6.28) was optimal. Students were given seven minutes to complete the measure.

Procedure

Prior to testing, high school students provided parental consent and personal assent and undergraduates provided personal consent. High school students and undergraduates received gift cards as compensation. Participants completed all measures during individual sessions of about 50 minutes; measures were administered in the same sequence for all participants, alternating longer and shorter measures to prevent participant fatigue. (Eye-tracking data were also collected for an additional set of CMR items not reported in the present manuscript.)

Data analysis

We conducted simple linear regressions using Mplus software version 7.11 (Muthen & Muthen, 1998–2013). For data analysis, all unanswered questions were scored as incorrect answers. We entered all variables in the regressions, and correlated each predictor with all other predictors. Nonsignificant correlations were then dropped and the models were re-run. In addition to the significance test on R^2, fit can be gauged with a statistic called the Standardized Root Mean Residual, which should be less than 0.08 to show evidence of a good model, together with a statistic called the Comparative Fit Index CFI, which should be greater than 0.95 (note these are the cutoffs for small samples; Hu & Bentler, 1999). Using Mplus software enabled us to simultaneously test coefficients for two dependent variables, thereby gaining statistical power. Even though Mplus is best known as Structural Equation Modeling software, we tested no mediators in our analyses.

Results

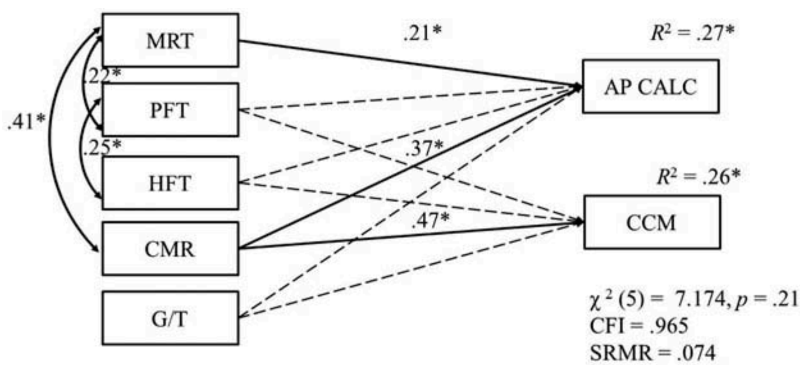
Correlations among and descriptive statistics on all measures are shown in Table 1. Recall that our goal was to use spatial skills measures and a measure of coordinating multiple representations, together with a measure of representational familiarity to predict scores on two measures of calculus proficiency (see Figure 1): a standardized calculus proficiency measure and a newly developed measure of conceptual understanding of calculus.

The regression accounted for a significant 27% of the variance in AP Calculus question scores (see Figure 1). The significant predictors of AP Calculus question scores were MRT (β = 0.21) and the CMR measure (β = 0.37). Among the spatial measures, MRT was a significant predictor of calculus proficiency after accounting for its shared variance with PFT, and for the shared variance of PFT with the Hidden Figures Test (HFT). Despite the correlation of PFT with AP Calculus question scores (r = 0.20) and of HFT with AP Calculus question scores (r = .27), only MRT emerged as a unique predictor in the regression with the other predictors entered. We return to this point in the discussion.

The regressions accounted for a significant 26% of the variance in Calculus Conceptual Knowledge scores (see Figure 1). For the Calculus Conceptual Measure, none of the spatial measures were significant predictors, despite correlations with HFT (r = 0.22), PFT (r = 0.20), and MRT (r = 0.27). We also return to this point in the discussion. CMR measure (β = 0.37) was a significant predictor of calculus conceptual knowledge.

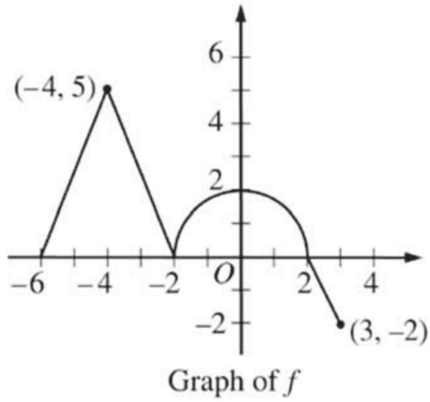
Discussion

In this brief study, we have shown that of the spatial skills measured, intrinsic, dynamic spatial skills (as measured by the MRT) uniquely predicts calculus proficiency as measured by released AP Calculus questions, but not calculus proficiency as measured by the Calculus Conceptual Measure. Using the terminology from Atit, Shipley, and Tikoff (2015), we can characterize the MRT as a test of rigid transformations (i.e., the block figure does not change shape) in 3D, the PFT as a test of nonrigid (i.e., the paper folds) transformations in 3D, and the HFT as a test of guided attention (2D). From this perspective, spatial skills with rigid transformations are most highly related to calculus proficiency as measured by the released AP calculus items. These are multistep calculation problems with derivatives and integrals that rely on knowledge of basic factual and procedural knowledge in calculus as well as flexible application of problem-solving strategies (see Figure 2). Although there is a slight risk that the



Note: Solid lines indicate significant paths; dashed lines indicate non-significant paths. MRT = Mental Rotations Test, PFT = Paper Folding Test, HFT = Hidden Figures Test, CMR = Coordinating Multiple Representations, G/T = Graph/Table basic comprehension, AP CALC = released Advanced Placement calculus test, CCM = Calculus Conceptual Measure

Figure 1. Results of the regressions.



The graph of the continuous function f, consisting of three line segments and a semicircle, is shown above. Let g be the function given by $g(x) = \int_{-2}^{x} f(t)\,dt$.

a) Find $g(-6)$ and $g(3)$.

b) Find $g'(0)$

c) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a horizontal tangent. Determine whether g has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

d) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a point of inflection. Explain your reasoning.

Figure 2. Sample item for spatially focused math instruction in calculus.

scores reflect speededness more than actual calculus proficiency, our item analysis suggests that almost all students did try to answer any question they felt they could, even questions at the end of the measure. One common teaching approach used by instructors and textbooks is to imagine the changing slope of the tangent line as it moves along a curve, or to imagine an accumulation function defined as an integral gaining or losing area as one endpoint is moved. Mentally imagining this set of steps may be similar to mentally imagining the rotation of the block figures in the MRT, and is less similar to the mental imagery needed for the PFT (i.e., folding) and the figure-ground discrimination tested by the HFT. Thus, it makes sense that of the predictors entered into the regression, MRT would significantly predict calculus proficiency.

Regarding coordinating multiple representations, both the released AP calculus questions and the Calculus Conceptual Measure included equation-plus-graph representations (the former used specific equations and the latter used abstract equations), so it makes sense that the CMR measure would be a significant predictor of both calculus outcomes. Why would spatial skills not predict scores for calculus conceptual knowledge? Recall that the measure for calculus conceptual knowledge required participants to demonstrate understanding of critical features in calculus problems, but did not involve multi-step problem-solving. Thus, we conclude that it is the learning of problem-solving in calculus where spatial skills with rigid transformations most likely play a role; by contrast, identifying, interpreting, and categorizing problem features seem like less likely leverage points.

Since spatial skills can be trained, including the specific skills underlying the MRT (Uttal et al., 2013, training mean $d = 0.44$ for intrinsic, dynamic spatial skills), one implication of our work is that spatial skills training might help some students succeed better in calculus. Since calculus is a critical subject at the high school and undergraduate levels for pursuing STEM degrees and careers, identifying spatial skills with rigid transformations as a malleable target for instruction has important practical implications for those interested in helping students remain in the STEM pipeline. Effective training can include practice on test-like items, lessons, or playing certain spatially-intensive games (e.g., Tetris; De Lisi & Wolford, 2002). Spatial skills tend to be slightly lower in female and inner-city elementary school students (Levine, Vasilyeva, Lourenco, Newcombe, & Huttenlocher, 2005), and these groups also struggle more with mathematics and with later calculus proficiency (Lubienski, 2002; Niederle & Vesterlund, 2010). Given that we have documented a relation between spatial skills and calculus proficiency, spatial skills interventions might serve a dual purpose toward increasing success among groups under-represented in science.

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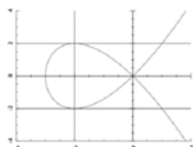
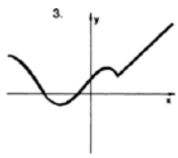
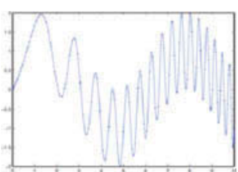
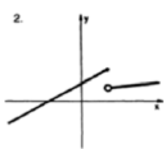
Appendix

Calculus Conceptual Knowledge Measure

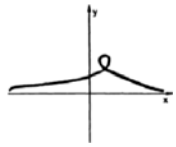
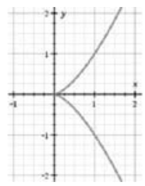
Directions: Solve each of the following problems, using any available space for scratchwork.

1. Is each of the following a graph of a function in the form of $y = f(x)$? Circle YES or NO.

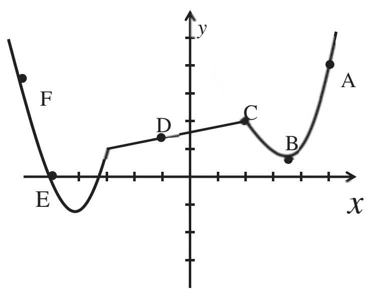
a.		YES	NO
b.		YES	NO
c		YES	NO
d.		YES	NO

(*Continued*)

(Continued).

e.		YES	NO
f.		YES	NO

2. For the function f whose graph is shown below, circle each labeled point(s) that satisfies the following conditions.



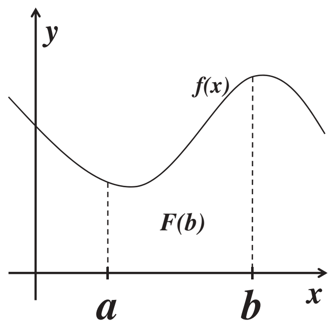
Circle all that apply

a.	$f'(x) = 0$	A	B	C	D	E	F
b.	$0 < f'(x) < 1$	A	B	C	D	E	F
c.	$f'(x) > 1$	A	B	C	D	E	F
d.	$f(x) = 0$	A	B	C	D	E	F
e.	$f'(x) < 0$	A	B	C	D	E	F
f.	$f'(x) < 1$	A	B	C	D	E	F
g.	$f'(x)$ is not defined	A	B	C	D	E	F

3. Which information would you use to determine each of the following? (Circle all that apply.)

a.	If f has a critical point at $x = 3$	f	f'	f''
b.	The zeros of f	f	f'	f''
c.	If the graph of f has an inflection point at $x = -1$	f	f'	f''
d.	Intervals on which f is decreasing	f	f'	f''

4. What information would you need to know about the function, $f(x)$(shown below), in order to determine each of the following? (Mark an X for each that applies.)



	An anti-derivative g exists	Whether it is even or odd	It is defined for all real values	It is continuous over a closed interval	It is differentiable on an open interval	Its points of inflection
a. If there exists a point, c, on *f(x)* such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ (If the **Mean Value Theorem** for derivatives applies)						
b. If the definite integral of the derivative of *f(x)* is given by $\int_{a}^{b} f(x)dx = g(b) - g(a)$, where g is the antiderivative (If the **First Fundamental Theorem of Calculus** applies)						
c. If the area *F(b)* is given by $F(b) = \int_{b}^{a} f(x)dt$ (If the **Second Fundamental Theorem of Calculus** applies)						

5. State whether each of the following is an accurate statement about limits. (Circle YES or NO.)

a.	The limit of a function $f(x)$ at a given value of x may not exist, even though $f(x)$ is defined at x	YES	NO
b.	In order to determine the limit of $f(x)$, the formula for $f(x)$ must be given	YES	NO
c.	The limit of a function $f(x)$ at a given value of x may be infinite	YES	NO
d.	If $f(x)$is continuous at a given value of x, it has a limit for that value of x	YES	NO
e.	The value of the limit of $f(x)$at the point $f(a)$ is the same as the value of $f(a)$	YES	NO