Linear Combinations

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Outline

- Motivation
- Population
- Estimation
- Example
- Two sets of Variables (and more linear algebra)

Reading: Johnson & Wichern pages 75–75, 149–192

Motivation

- A major tool for data reduction & insight into why reject H_o regarding μs.
- Definition: A Linear Combination of p (random) variables $X_1, X_2, \cdots, X_p$ is $a_1X_1 + a_2X_2 + \cdots + a_pX_p$.
  Let
  $$\boldsymbol{a}' = (a_1, a_2, \cdots, a_p) \qquad \text{and} \qquad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}$$
  A linear combination in terms of vector operations is
  $$\boldsymbol{a}'\boldsymbol{X}_{p\times 1}$$
- The linear combination $\boldsymbol{a}'\boldsymbol{X}$ is itself a random variable.

Mean of Linear Combination

The mean of $\mathbf{a}'\mathbf{X}$ is

$$\begin{aligned} E(\mathbf{a}'\mathbf{X}) &= E(a_1X_1 + a_2X_2 + \cdots + a_pX_p) \\ &= E(a_1X_1) + E(a_2X_2) + \cdots + E(a_pX_p) \\ &= a_1E(X_1) + a_2E(X_2) + \cdots + a_pE(X_p) \\ &= a_1\mu_1 + a_2\mu_2 + \cdots + a_p\mu_p \\ &= \mathbf{a}'\boldsymbol{\mu} \end{aligned}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}$$

Know this: $(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\boldsymbol{\mu}$

Variance of a Linear Combination

$$\begin{aligned}
\text{var}(\boldsymbol{a}'\boldsymbol{X}) &= \text{var}(a_1X_1 + a_2X_2 + \cdots + a_pX_p) \\
&= a_1^2\text{var}(X_1) + a_2^2\text{var}(X_2) + \cdots + a_p^2\text{var}(X_p) \\
&\quad + a_1a_2\text{cov}(X_1, X_2) + \cdots + a_{p-1}a_p\text{cov}(X_{p-1}, X_p) \\
&= \sum_{i=1}^{p} a_i^2\sigma_{ii} + \sum_{i}\sum_{k}\limits_{i\neq k} a_ia_k\sigma_{ik} = \sum_{i}\sum_{k} a_ia_k\sigma_{ik}
\end{aligned}$$

For $p = 2$,

$$\begin{aligned}
\text{var}(\boldsymbol{a}'\boldsymbol{X}) &= a_1^2\sigma_{11} + 2a_1a_2\sigma_{12} + a_2^2\sigma_{22} \\
&= (a_1, a_2)\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\
&= \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a}
\end{aligned}$$

A Quadratic Form: For any p, $\text{var}(\boldsymbol{a}'\boldsymbol{X}) = \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a}$

Results if we add a constant

If we add a constant to the linear combination,

$$a_1 X_1 + a_2 X_2 + \cdots + a_p X_p + c = \boldsymbol{a}'\boldsymbol{X} + c$$

Then

Mean: Changes location

$$E(\boldsymbol{a}'\boldsymbol{X} + c) = \boldsymbol{a}'E(\boldsymbol{X}) + c = \boldsymbol{a}'\boldsymbol{\mu} + c$$

Variance: No change

$$\text{var}(\boldsymbol{a}'\boldsymbol{X} + c) = \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a}$$

Multiple Linear Combinations

Sometimes one just isn't enough...

Suppose that there are q linear combinations of the p random variables,

$$\begin{aligned} y_1 &= a_{11}X_1 + a_{12}X_2 + \cdots + a_{1p}X_p \\ y_2 &= a_{21}X_1 + a_{22}X_2 + \cdots + a_{2p}X_p \\ \vdots & \quad \vdots \\ y_q &= a_{q1}X_1 + a_{q2}X_2 + \cdots + a_{qp}X_p \end{aligned}$$

$$\boldsymbol{y}_{q\times 1} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{qp} & a_{q2} & \cdots & a_{qp} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} = \boldsymbol{A}_{q\times p}\boldsymbol{X}_{p\times 1}$$

Mean and Covariance for $\boldsymbol{Y}(q \times 1)$

$$\boldsymbol{y}_{q\times 1} = \boldsymbol{A}_{q\times p}\boldsymbol{X}_{p\times 1}$$

The Mean vector for $\boldsymbol{y}$,

$$\mu_{\boldsymbol{y}} = E(\boldsymbol{y}) = E(\boldsymbol{AX}) = \boldsymbol{A}E(\boldsymbol{X}) = \boldsymbol{A}\mu_{\boldsymbol{X}}$$

The Covariance matrix for $\boldsymbol{y}$,

$$\Sigma_{\boldsymbol{y}} = \text{cov}(\boldsymbol{y}) = \text{cov}(\boldsymbol{AX}) = \boldsymbol{A\Sigma A}'$$

These are 2 very important results!

$$\mu_{\boldsymbol{y}} = \boldsymbol{A}\mu_{\boldsymbol{X}}$$

$$\Sigma_{\boldsymbol{y}} = \boldsymbol{A\Sigma_{X}A}'$$

Kind of an Example

Let $\boldsymbol{X}$ be a random vector

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad \text{with} \quad \boldsymbol{\mu_X} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Suppose we're interested in

- The average: $Y_1 = (1/3)(X_1 + X_2 + X_3)$
- Contrast: $Y_2 = (X_1 - X_2)$

$$\begin{aligned} \boldsymbol{\mu_y} = \begin{pmatrix} \mu_{y_1} \\ \mu_{y_2} \end{pmatrix} &= \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \end{pmatrix} E \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \\ &= \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(\mu_1 + \mu_2 + \mu_3) \\ \mu_1 - \mu_2 \end{pmatrix} \end{aligned}$$

Kind of an Example

...and the covariance matrix of $\boldsymbol{Y}$,

$$\boldsymbol{\Sigma_y} = \boldsymbol{A\Sigma_x A'} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1/3 & 1 \\ 1/3 & -1 \\ 1/3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3}(\sigma_{11}+\sigma_{21}+\sigma_{31}) & \frac{1}{3}(\sigma_{12}+\sigma_{22}+\sigma_{32}) & \frac{1}{3}(\sigma_{13}+\sigma_{23}+\sigma_{33}) \\ (\sigma_{11}-\sigma_{21}) & (\sigma_{12}-\sigma_{22}) & (\sigma_{13}-\sigma_{23}) \end{pmatrix} \begin{pmatrix} 1/3 & 1 \\ 1/3 & -1 \\ 1/3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9}(\sigma_{11}+\sigma_{22}+\sigma_{33}+2(\sigma_{12}+\sigma_{13}+\sigma_{23})) & \frac{1}{3}(\sigma_{11}-\sigma_{22}+\sigma_{13}-\sigma_{23}) \\ \frac{1}{3}(\sigma_{11}-\sigma_{22}+\sigma_{13}-\sigma_{23}) & (\sigma_{11}+\sigma_{22}-2\sigma_{12}) \end{pmatrix}$$

Estimation of μ_y

1^{st} just one linear transformation (combination, composite, etc)

Let $\boldsymbol{a}' = (a_1, a_2, \cdots, a_p)$ and $\boldsymbol{x}' = (x_1, x_2, \cdots, x_p)$ and

$$y = \boldsymbol{a}'\boldsymbol{x} = \sum_{i=1}^{p} a_i x_i$$

If we have n cases, the sample observation of the linear transformation for the j^{th} case (individual) is y_j,

$$\boldsymbol{a}'\boldsymbol{x}_j = a_1 x_{j1} + a_2 x_{j2} + \cdots + p x_{jp} \quad \text{for } j = 1, \ldots, n$$

Sample Mean of y :

$$\begin{aligned} \bar{y} &= \frac{1}{n}(y_1 + y_2 + \ldots + y_n) \\ &= \frac{1}{n}(\boldsymbol{a}'\boldsymbol{x}_1 + \boldsymbol{a}'\boldsymbol{x}_2 + \cdots + \boldsymbol{a}'\boldsymbol{x}_n) \\ &= \boldsymbol{a}'\frac{1}{n}\sum_{j=1}^{n} x_j = \boldsymbol{a}'\bar{\boldsymbol{x}} \end{aligned}$$

Estimation of var(y)

Sample variance of y:

$$\begin{aligned}
\text{var}(y) &= \frac{1}{n-1}\sum_{j=1}^{n}(y_j - \bar{y})^2 \\
&= \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{a}'\mathbf{x}_j - \mathbf{a}'\bar{\mathbf{x}})^2 \\
&= \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{a}'\mathbf{x}_j - \mathbf{a}'\bar{\mathbf{x}}) \underbrace{(\mathbf{a}'\mathbf{x}_j - \mathbf{a}'\bar{x})'}_{(\mathbf{a}'(\mathbf{x}_j - \bar{\mathbf{x}}))' = ((\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{a})'} \\
&= \frac{1}{n-1}\sum_{j=1}^{n}\mathbf{a}'(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{a} \\
&= \mathbf{a}'\left(\frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'\right)\mathbf{a} \\
&= \mathbf{a}'\mathbf{S}\mathbf{a}
\end{aligned}$$

Estimation of $\bar{y}$ and var(y)

$\boldsymbol{a}'\bar{\boldsymbol{x}}$ is the sample analog to $\boldsymbol{a}'\boldsymbol{\mu}$

$\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ is the sample analog to $\boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a}$

Let's add a 2^{nd} linear combination of the vector $\boldsymbol{X}$:

$$Z = b_1X_1 + b_2X_2 + \cdots + b_pX_p = \boldsymbol{b}'\boldsymbol{X} = \sum_{i=1}^{p} b_iX_i$$

The j^{th} sample observation on variable Z is

$$z_j = \boldsymbol{b}'\boldsymbol{x}_j = b_1x_{j1} + b_2x_{j2} + \cdots + b_px_{jp}$$

Applying above results gives us

Sample mean: $\bar{z} = \boldsymbol{b}'\bar{x}$

Sample variance: $s_z^2 = \boldsymbol{b}'\boldsymbol{S}\boldsymbol{b}$

What's the covariance between y and z?

Sample Covariance

Between y and z

$$\begin{aligned}
\text{cov}(y,z) &= \frac{1}{n-1}\sum_{j=1}^{n}(y_j-\bar{y})(z_j-\bar{z}) \\
&= \frac{1}{n-1}\sum_{j=1}^{n}(\boldsymbol{a}'\boldsymbol{x}_j-\boldsymbol{a}'\bar{\boldsymbol{x}})(\boldsymbol{b}'\boldsymbol{x}_j-\boldsymbol{b}'\bar{\boldsymbol{x}}) \\
&= \frac{1}{n-1}\boldsymbol{a}'\left(\sum_{j=1}^{n}(\boldsymbol{x}_j-\bar{\boldsymbol{x}})(\boldsymbol{x}_j-\bar{\boldsymbol{x}})'\right)\boldsymbol{b} \\
&= \boldsymbol{a}'\left(\frac{1}{n-1}\sum_{j=1}^{n}(\boldsymbol{x}_j-\bar{\boldsymbol{x}})(\boldsymbol{x}_j-\bar{\boldsymbol{x}})'\right)\boldsymbol{b} \\
&= \boldsymbol{a}'\boldsymbol{S}\boldsymbol{b}
\end{aligned}$$

Example with Numbers

Suppose we have $n = 3$ and $p = 3$, and the data matrix is

$$\boldsymbol{X} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ 1 & 4 & 3 \end{pmatrix}$$

And we want to find the two linear combinations

$$\boldsymbol{b}'\boldsymbol{x} = b_1 x_1 + b_2 x_2 + b_3 x_3$$

where $\boldsymbol{b}' = (1, 1, 1)$ and $\boldsymbol{x}$ is a row of $\boldsymbol{X}$ written as a column vector, and

$$\boldsymbol{c}'\boldsymbol{x} = c_1 x_1 + c_2 x_2 + c_3 x_3$$

where $\boldsymbol{c} = (-2, 1, 1)$.

Observations on "new" variables

Observations on these two "new" variables equal

$$\boldsymbol{b}'\boldsymbol{X}' = (1,1,1)\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 3 \end{pmatrix} = (7,7,8)$$

$$\boldsymbol{c}'\boldsymbol{X}' = (-2,1,1)\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 3 \end{pmatrix} = (4,4,5)$$

In terms of single matrix formula,

$$\boldsymbol{A}\boldsymbol{X}' = \begin{pmatrix} \boldsymbol{b}' \\ \boldsymbol{c}' \end{pmatrix}\boldsymbol{X}' = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 7 & 8 \\ 4 & 4 & 5 \end{pmatrix}$$

Mean and Covariance for x's

The means on the original variables are

$$\bar{x}_1 = 1, \quad \bar{x}_2 = 3, \quad \text{and } \bar{x}_3 = 3.333$$

Sample means of new variables computed using the scalar formula with the observations on the new variables (on the left) and matrix formula using the observations on the old variables (on the right):

$$\begin{pmatrix} \frac{1}{3}(7+7+8) \\ \frac{1}{3}(4+4+5) \end{pmatrix} = \begin{pmatrix} 7.33 \\ 4.33 \end{pmatrix} = \boldsymbol{A\bar{x}} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3.33 \end{pmatrix} = \begin{pmatrix} 7.3 \\ 4.3 \end{pmatrix}$$

Sample Variances & Covariance for x's

Sample variances of the old variables:

$$\begin{aligned} s_{11} &= 0 \\ s_{22} &= \frac{1}{2}\left[(2-3)^2 + (3-3)^2 + (4-3)^2\right] = 1 \\ s_{33} &= \frac{1}{2}\left[(4-3.33)^2 + (3-3.33)^2 + (3-3.33)^2\right] = \frac{1}{3} \end{aligned}$$

Sample covariances between the old variables:

$$\begin{aligned} s_{12} &= \frac{1}{2}\left[0(2-3) + 0(3-3) + 0(4-3)\right] = 0 \\ s_{13} &= \frac{1}{2}\left[0(4-3.33) + 0(3-3.33) + 0(3-3.33)\right] = 0 \\ s_{23} &= \frac{1}{2}\left[(2-3)(4-3.33) + (3-3)(3-3.33) + (4-3)(3-3.33)\right] \\ &= \frac{1}{2}(-.67 + 0 - .33) = -\frac{1}{2} \end{aligned}$$

Sample Covariance Matrices

The sample variance-covariance matrix of the x's variables is

$$\boldsymbol{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

For the new variables (the y's):

$$\begin{aligned} \boldsymbol{ASA'} &= \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{2} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{aligned}$$

Note regarding var(y)'s and cov(y_i, y_2)

The results on the previous slide are the same if we had used scalar formulas with the values of the y_{j1} and y_{j2}:

$$\begin{aligned}
s_{11} &= \frac{1}{2}[(7-7.33)^2+(7-7.33)^2+(8-7.33)^2] = \frac{1}{3} \\
s_{22} &= \frac{1}{2}[(4-4.33)^2+(4-4.33)^2+(5-4.33)^2] = \frac{1}{3} \\
s_{12} &= \frac{1}{2}[(7-7.33)(4-4.33)+(7-7.33)(4-4.33) \\
&\quad +(8-7.33)(5-4.33) = \frac{1}{3}
\end{aligned}$$

It was just by coincidence that all the values are 1/3

. . . bad choice of "data" on my part.

Two Sets of Linear Combinations

If we have one set of q linear combinations and another set of r linear combinations of random variables $\boldsymbol{X}$, then

$$\boldsymbol{Y}_{q\times 1} = \boldsymbol{A}_{q\times p}\boldsymbol{X}_{p\times 1} \qquad \boldsymbol{Z}_{r\times 1} = \boldsymbol{B}_{r\times p}\boldsymbol{X}_{P\times 1}$$

$$E(\boldsymbol{Y}) = \boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{X}} \qquad E(\boldsymbol{Z}) = \boldsymbol{B}\boldsymbol{\mu}_{\boldsymbol{X}}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{\Sigma}_{\boldsymbol{X}}\boldsymbol{A}' \qquad \boldsymbol{\Sigma}_{\boldsymbol{z}} = \boldsymbol{B}\boldsymbol{\Sigma}_{\boldsymbol{X}}\boldsymbol{B}'$$

$$\boldsymbol{\Sigma}_{\boldsymbol{yz}} = \boldsymbol{A}\boldsymbol{\Sigma}_{\boldsymbol{X}}\boldsymbol{B}'$$

This is just an application of what we already did. . .

More Linear Algebra: Partitioned Matrices

Let matrix $\boldsymbol{A}_{k\times 1}$ be Partitioned into two parts as follows

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{qp} \\ \hline a_{q+1,1} & a_{q+1,2} & \cdots & a_{q+1,p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{q+r,1} & a_{q+r,2} & \cdots & a_{q+r,p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{1,(q\times p)} \\ \hline \boldsymbol{A}_{2,(r\times p)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{q\times p} \\ \hline \boldsymbol{B}_{r\times p} \end{pmatrix}$$

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_q \\ Y_{q+1} \\ \vdots \\ Y_{q+r} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Y}_{1,(q\times 1)} \\ \hline \boldsymbol{Y}_{2,(r\times 1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Y}_{(q\times 1)} \\ \hline \boldsymbol{Z}_{(r\times 1)} \end{pmatrix}$$

Partitioned mean and covariance matrices

$$\boldsymbol{\mu}_{(q+r)\times 1} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \mu_{q+1,1} \\ \vdots \\ \mu_{q+r} \end{pmatrix} = \left(\frac{\boldsymbol{\mu}_{1,(q\times 1)}}{\boldsymbol{\mu}_{2,(r\times 1)}} \right) = \left(\frac{\boldsymbol{\mu}_{Y,(q\times 1)}}{\boldsymbol{\mu}_{Z,(r\times 1)}} \right)$$

$$\boldsymbol{\Sigma}_{(q+r),(q+r)} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1,q+r} \\ \vdots & \ddots & \ldots & \vdots \\ \sigma_{q1} & \sigma_{q2} & \cdots & \sigma_{q,q+r} \\ \sigma_{q+1} & \sigma_{q+1,2} & \cdots & \sigma_{q+1,q+r} \\ \vdots & \ddots & \ldots & \vdots \\ \sigma_{q+r,1} & \sigma_{q+r,2} & \cdots & \sigma_{q+r,q+r} \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{yy,(q\times q)} & \boldsymbol{\Sigma}_{yz,(q\times r)} \\ \hline \boldsymbol{\Sigma}_{zy,(r\times q)} & \boldsymbol{\Sigma}_{zz,(r\times r)} \end{array} \right)$$

where $\boldsymbol{\Sigma}_{yz} = \boldsymbol{\Sigma}'_{zy}$.

Operations on Partitioned Vectors & Matrices

Treat each sub-matrix (vector) as an element of a matrix (vector).

For example

$$\begin{pmatrix} \mathbf{A}_{q\times p} \\ \hline \mathbf{B}_{r\times p} \end{pmatrix} \mathbf{X}_{p\times 1} = \begin{pmatrix} (\mathbf{AX})_{q\times 1} \\ \hline (\mathbf{BX})_{r\times 1} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_{q\times 1} \\ \hline \mathbf{Z}_{r\times 1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}_{q\times p} \\ \hline \mathbf{B}_{r\times p} \end{pmatrix} \boldsymbol{\mu}_{\mathbf{X}(p\times 1)} = \begin{pmatrix} (\mathbf{A}\boldsymbol{\mu})_{q\times 1} \\ \hline (\mathbf{B}\boldsymbol{\mu})_{r\times 1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{\mathbf{Y},(q\times 1)} \\ \hline \boldsymbol{\mu}_{\mathbf{Z},(r\times 1)} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}_{q\times p} \\ \mathbf{B}_{r\times p} \end{pmatrix} \boldsymbol{\Sigma}_{p\times p} \left(\mathbf{A}'_{p\times q} \mid \mathbf{B}'_{p\times q} \right) = \left(\begin{array}{c|c} \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' & \mathbf{A}\boldsymbol{\Sigma}\mathbf{B}' \\ \hline \mathbf{B}\boldsymbol{\Sigma}\mathbf{A}' & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{yy} & \boldsymbol{\Sigma}_{yz} \\ \hline \boldsymbol{\Sigma}_{zy} & \boldsymbol{\Sigma}_{zz} \end{array}\right),$$

which gives us the results for linear combinations for 2 sets of variables.